LOGISTIC AND THE REDUCTION OF MATHEMATICS TO LOGIC.

IN the year 1901 we find in an article by Bertrand Russell:[1] "The nineteenth century which prides itself upon the invention of steam and evolution, might have derived a more legitimate title to fame from the discovery of pure mathematics....One of the chiefest triumphs of modern mathematics consists in having discovered what mathematics really is.... Pure mathematics was discovered by Boole in a work which he called *The Laws of Thought*.... His work was concerned with formal logic, and this is the same thing as mathematics."

Also in Keyser's address[2] we find: "....the two great components of the critical movement, though distinct in origin and following separate paths, are found to converge at last in the thesis: Symbolic Logic is Mathematics, Mathematics is Symbolic Logic, the twain are one."

On the other hand we find Poincaré[3] saying after his various successful attacks on logistic: "Logistic has to be made over, and one is none too sure of what can be saved. It is unnecessary to add that only Cantorism and Logistic are meant, true mathematics, those which serve some useful purpose, may continue to develop according to their own principles without paying any attention to the tempests raging without them, and they will pursue step by step

[1] *International Monthly*, 1901.

[2] *Columbia University Lectures.*

[3] *Science et méthode*, p. 206.

their accustomed conquests which are definitive and which they will never need to abandon."

What then is this logistic which made such extravagant claims in 1901 and in 1909 was dead? In order to understand it we must go back to the third century B. C. when Aristotle was developing the study usually called logic. The logic of Aristotle is well enough defined when it is called the logic of classes. A class may be defined in the following terms: Let us suppose that we start with a proposition about some individual, as for example, "8 is an even number," or as another case, "Washington crossed the Delaware." If now we remove the subject and substitute the empty form x, we shall have the statements: "x is an even number, x crossed the Delaware," which are called propositional functions, from analogy to mathematical functions. In this case the functions have but one variable or empty term, x. If we let x run through any given range of objects, the resulting statements will be some true, some false, some senseless. Those that are true or false constitute a list of propositions. For example we may say: "6 is an even number, 9 is an even number, this green apple is an even number," the first a true proposition, the second a false proposition, the third an absurdity. So I might say: "Washington crossed the Delaware, the Hessians crossed the Delaware, the North Pole crossed the Delaware," which are respectively true, false, and absurd, the first two cases being propositions. The propositional function with one variable is called a concept. The individuals that may be put into the empty term (which may be any word of the statement), the variable, and yield true propositions constitute the class of the concept. Thus the class of even numbers consists of a certain endless set or range of individuals, the class of presidents of the United States a certain set of a few individuals, the President of the United States of one individual, and the class of simple

groups of odd order may consist of no individuals at all. The individuals of a class may not be known, for instance the daily temperatures at the North Pole, or the odd perfect numbers. It is practically impossible to ascertain the individuals in the first class, and there may not be any in the second class mentioned. In case it can be shown that a class has no individuals it is called a null-class. It should be noted carefully that the individuals do not define the class, but conversely the class defines the individuals. The same individuals may be defined by one or more classes. Nor is the relation of a member of a class to the class the same as the relation of a subclass to the class. For instance we may discuss the class of numbers which are either multiples of 5 or give a remainder 1 when divided by 5. Now the class of fourth powers of integers are all either divisible by five or give 1 for remainder. Hence the fourth powers constitute a subclass of the first class mentioned. But of any one fourth power, as 81, say, we cannot assert that 81 has the property of being divisible by 5 or of giving a remainder one, and its relation to the class is different from the relation of the subclass to the class. A subclass is said to be included in the class, not to be a member of it. This difference was first pointed out by Peano[4] and was not known to Aristotle. The two relations are indicated by the symbols ε and ·(·, for instance,

Roosevelt ε presidents of the United States,
some square roots ·(· irrationals.

The symbol of a class is the inverted ε, ɜ, for instance

x ɜ divisor of 288,

read "the class of divisors of 288." It is evident that a class is not a class of classes, for the latter is a class of propositional functions of one variable, the former a class of individuals.

[4] *Formulaire de mathématique,* Vol. I.

Aristotle not only studied classes, with schemes for definition and subdivision of classes, but he introduced the syllogism as a means of reasoning. The syllogism is a succession of three statements of the inclusions of classes; in formal statement, Greek letters denoting classes,

$$\alpha \cdot (\cdot \beta, \beta \cdot (\cdot \gamma, \text{ then } \alpha \cdot (\cdot \gamma.$$

For example, Pascal's theorem is true of any conic, every circle is a conic, whence Pascal's theorem is true of every circle. For an individual circle we should have a different type of syllogism, a distinction not noted by Aristótle, namely

$$x \,\varepsilon\, \alpha, \ \alpha \cdot (\cdot \beta, \text{ then } x \,\varepsilon\, \beta.$$

For instance, Pascal's theorem is true of circles, this figure is a circle, thence Pascal's theorem is true for this individual circle.

Logic rested with the Aristotelian development for many centuries, and was supposed to be perfect. The regeneration of the subject has been ascribed to Leibniz, because he hoped to see a universal symbolism which would enable the complete determination of all the consequences of a given set of premises to be easily carried out, just as mathematical formulas enable us to solve large classes of problems. This was his Universal Characteristic. But it was reserved for a later day to bring to light the symbolic logic, and we may pass at once to Boole[5] and the nineteenth century. We shall find however in the invention of Boole and his successors not the discovery of mathematics but the mathematicising of logic. The mind again devises new forms for its own use, new ideas by which to attack its problems.

Boole used letters to express classes, the conjunction

[5] *The Mathematical Analysis of Logic*, 1847; *An Investigation of the Laws of Thought*, 1854.

of two letters indicating the largest common subclass, and the formal addition of two letters the smallest common superclass. Then the laws of logic are stated by the formal equations

> $a = aa$, (identity); $a + ab = a$, $a(a + b) = a$, (absorption); $ab = ba$, $a + b = b + a$, (commutation); $aa = a$, $a + a = a$, (tautology); $ab = aba$, $a = a(a + b)$, (simplification); $a = ab$, $a = ac$, then $a = abc$, (composition).

He introduced two constants called logical constants, represented by 1 and 0, with the meaning for 1, the minimum superclass of all classes considered, the logical universe; and for 0, the greatest common subclass of all classes, the null-class, or class of impossibilities. It is understood that if a class is considered, the negative of the class is also under consideration, represented by a'. If only one class is considered then $1 = a + a'$. If two are considered $1 = ab + ab' + a'b + a'b'$, etc. It is evident that

$$1a = a,\ 1 + a = 1,\ 0a = 0,\ 0 + a = a.$$

The invention of these notions which seem simple enough now was a great advance over the logic of Aristotle. It suggested for example the use of $1 - a$ for a', with the formulas corresponding to algebra

$$a(1 - a) = 0,\ 1 = a + (1 - a),$$

the laws of contradiction and excluded middle. Any class may be dichotomized in the form

$$x = ax + a'x = abx + ab'x + a'bx + a'b'x = \ldots$$

If x is a subclass of a we indicate it by the equations

$$x = ax \text{ or } xa' = 0.$$

The syllogism takes the very simple form

$$a = ab,\ b = bc, \text{ then } a = abbc = abc = ac.$$

We have thus invented a simple algebra which, with the one principle of substitution of any expression for a letter which the letter formally equals, and the reduction of all expressions by the laws of the algebra, enables us to solve easily all the questions of the older logic. Jevons[6] has stated the rule for doing this very simply: "State all premises as null-classes, construct all necessary subclasses by dichotomy, erase all combinations annulled by the premises, and translate the remaining expressions, by condensation, into the simplest possible equivalent language."

Boole however made a further most important discovery, that there is a nearly perfect analogy between the calculus of classes and the calculus of propositions. That is, we may interpret the symbols used above as representing propositions, under the following conventions. If a is a proposition, a' is the contradictory proposition, ab a proposition equivalent to the joint assertion of a and b, $a+b$ the assertion of either a or b or both, 1 a proposition asserting one at least of all the propositions and their contradictories under consideration, and 0 a proposition asserting all the propositions and their contradictories simultaneously, that is, 1 asserts consistency, 0 inconsistency. A series of formal laws may now be written out and interpreted similar to those for classes. The syllogism, for instance, is the same,

$a=ab$, $b=bc$, then $a=ac$; or in equivalent forms,
$ab'=0$, $bc'=0$, then $ac'=0$.

That is, if the assertion of a is equivalent to also asserting b, and if the assertion of b is equivalent to also asserting c, then the assertion of a is equivalent to the assertion of c. We may reduce the whole scheme of deduction as before to a system of terms which are the expansions of the possible list of simultaneous assertions, the premises annulling

[6] *Principles of Science,* also *Pure Logic.* See also *Studies in Deductive Logic.* Also Couturat, *Algèbre de la logique* (*Algebra of Logic,* translated by Robinson).

certain of these, and those remaining furnishing the conclusions. We should however note carefully that what we arrive at in this manner are not truths or falsehoods but consistencies and inconsistencies. That is to say, we do not prove anything to be true or false by the logic of propositions, we merely exhibit the assertions or classes with which it is consistent or compatible, or the reverse. In this sense only does logic furnish proof. It is obvious however that many new combinations of the symbols used are possible by these methods, and thus it is easy to ascertain the consistency of assertions that would not otherwise occur to us. While the premises evidently are the source of the conclusions, the conclusions are not the premises, and on the one hand the transition from the one to the other is made most easily by these methods, and on the other hand the conclusions are new propositions consistent with the premises. A simple example will show what is meant:

If a implies a', then a is 0; for if $aa=0$, at once $a=0$.

$$\text{Conv. if } a'a'=0,\ a'=0,\ a=1.$$

That is, a proposition which implies its contradictory is not consistent.

It should be noted that the calculus of propositions is not wholly parallel to the calculus of classes. This is shown particularly in the application of a certain axiom, as follows:

$(a\varepsilon \text{ true})=a$ Ax. $a'=(a'\varepsilon \text{ true})=(a\varepsilon f)$. This is absurd for the logic of classes, since $a=1$ is a proposition and not reducible to a class.

A useful form for implication is

$$(a \text{ implies } b)=(a'+b=1).$$

The next advance was due to C. S. Peirce,[7] who devised

[7] *Mem. Amer. Acad. Arts and Sciences,* N. S., IX, 1870, pp. 317-378.

the logic of relatives, in which the propositional function with two variables appears, and which may readily be generalized into the propositional function with any number of variables, giving binary, ternary, and then n-ary relatives. As simple examples we may omit individuals that satisfy the proposition: A is the center of the circle c, arriving at the propositional function: x is the center of y; or another example with four variables is found in: x is the harmonic of y as to u and v. The calculus of the logic of relations is obviously much more complicated than the previously known forms of symbolic logic. While some of the theorems and methods of the calculus of classes and propositions may be carried over to the calculus of relations, there are radical differences. For instance the relation $x\mathrm{R}y$ is the converse of the relation $y\mathrm{R}x$. These two relations are not identical unless R is symmetric. Again from $x\mathrm{R}y$, $y\mathrm{R}z$, we can infer $x\mathrm{R}z$ only if R is transitive. The ranges of a relation are the sets of individuals that satisfy the propositional function, when inserted for some one of the variables. The most complete development of these notions is to be found in Whitehead and Russell's *Principia Mathematica*. In the intoxication of the moment it was these outbursts of the mind that led Russell into the extravagant assertions he made in 1901. In the *Principia* there are no such claims. It should be noted too that the work of Whitehead in his *Universal Algebra* (1898) contained a considerable exposition of symbolic logic.

As soon as the expansion of logic had taken place Peano undertook to reduce the different branches of mathematics to their foundations and subsequent logical order, the results appearing in his *Formulario,* now in its fifth edition. In the *Principia* the aim is more ambitious, namely to deduce the whole of mathematics from the undefined or assumed logical constants set forth in the beginning. We

must now consider in a little detail this ambitious program and its outcome.

The basal ideas of logistic are to be found in the works of Frege, but in such form that they remained buried till discovered by Russell after he had arrived himself at the invention of the ideas independently. The fundamental idea is that of the notion of function extended to propositions. A propositional function is one in which certain of the words have been replaced by variables or blanks into which any individuals may be fitted. This isolation of the functionality of an assertion from the particular terms to which it is applied is a distinctly mathematical procedure, and entirely in line with the idea of function as used in mathematics. It enabled us above to define concept and relation, in a way, and it further makes quite clear in how great a degree mathematical theorems are about propositional functions and not about individuals. For instance, the statement, "If a triangle has a right angle it may be inscribed in a semicircle," merely means

> right-angled-triangularity as a property is inconsistent with non-inscribability-in-a-semicircle as a property.

In this mode of statement it is apparent to every one that a large part of mathematics is concerned with the determination of such consistencies or inconsistencies. That it is not wholly concerned with them however is also quite apparent. For example, the calculation of π can only be called a determination of the figures consistent with certain decimal positions by a violent straining of the English language. And again, the determination of the roots of an equation is a determination of the individuals which will satisfy a given propositional function, and not a determination of the other functions consistent or inconsistent with that first function. There is a difference well known to

any mathematician between the properties of the roots of a quadratic equation and the properties of quadratic functions of x. Again, the analysis of the characteristics of a given ensemble is a determination of the essential constituents of the propositional function whose roots are the individuals of the ensemble. Operators considered as such are not propositional functions, and neither are hypernumbers. It has been made quite clear, we hope, in what precedes, that much of the mathematician's work consists in building up constructions, and determining their characteristics, and not in considering the functions of which such constructions might be roots. There is a difference between the two assertions

> $2+3=5$ and, If 2 is a number, and if 3 is a number, and if 2 and 3 be added, then we shall produce a number which is 5.

We find the difference well marked in the logistic deduction of the numbers one and two. The deduction is as follows:

Let us consider the propositional functions

"$x \,\varepsilon\, \varphi_1$ has only roots such that they cannot be distinguished," as likewise $x \,\varepsilon\, \varphi_2, \ldots$ For instance let $()=6$, of which the roots are $4+2$, 2×3, $12/2, \ldots$ which are all indistinguishable in this propositional function. So also $()=9$, $()=4/3 \ldots$. Then if we call these propositions similar, in that each *has indistinguishable roots,* we may consider next the propositional function $p \; sim \; ()=6$, where p is a variable proposition, which however is distinguished by the character of indistinguishable roots. We may now define the number 1 as the functionality in this functional proposition. That is to say, 1 is a property of propositional functions—namely, that of uniqueness of their roots. In mathematical language we might say: The character which is common to all equations of the form $(x-a)^n=0$, is called *one,* thus defining *one.* Now while it is true perhaps,

that to seize upon equations with one root as cases in which oneness appears, is a valid way to arrive at *one,* nevertheless it is not at all different from any other case in which oneness occurs, as in selecting one pencil from a pile of pencils. In a like manner two is defined as the common property of propositional functions which are relations with a twofold valence, that is, admit two series of roots, the series in each case consisting of indistinguishable individuals. The truth of the matter is that the definitions given are merely statements in symbolic form of cases in which the number one or the number two appears. The two numbers have in no wise been deduced, any more than a prestidigitator produces a rabbit from an empty hat, but they have first been caught, then simply exhibited in an iron cage. The fact that functions are useful things we cheerfully admit, but that everything is reducible to logical functions we do not admit. The arithmetic of 2 and 1 was known long before logistic.

Another notion introduced by logistic is that of truth and truth-value. In no place are either of these terms made clear, nor are they defined. They are qualities of *propositions,* that is propositional functions which have had individuals inserted for the variables. For example, if I consider the propositional function x is right-angled, and then for x insert respectively the triangle ABC, the parallelogram S, this pink color, I have the propositions ABC is right-angled, the parallelogram S is right-angled, this pink color is right-angled. The first of these is said to have the truth-value *truth,* the second the truth-value *false,* the third has the value *absurd,* which is not a truth-value. The first two assertions are then propositions, the third is not a proposition. Much is made of the idea of truth-value, but practically it amounts only to saying that an assertion is a proposition only when it can be labeled with one of two given labels. If any other label is neces-

sary it is not a proposition and not within the region of logistic. So far as really used in logistic these labels are neither more nor less than labels of consistency and inconsistency. They do not refer in any way to objective truth. Thus if we start with the postulates of Euclidean geometry we arrive at certain propositions, as, "triangle ABC has the sum of its angles equal to two right angles." This proposition is not to be tagged as true, but merely as consistent with the premises we started with. The determination of the primitive truth of the premises is not possible by logistic at all. The whole of science is of this character, the truth of the conclusions of science being only probable, not certain, although the reasoning is valid. Science draws its validity from the agreement of all its conclusions with experience. In the same way the conclusions of mathematics are consistent under our notions of consistency, but neither true nor false on account of the reasoning. And this is all that Russell is privileged to say when he asserts that "mathematics is the science in which we do not know whether the things we talk about exist nor whether our conclusions are true." From the results of logistic we certainly do not know either of these things. We merely know that if they exist, and if the premises are true, then the conclusions are true provided the processes of logistic can give true conclusions.

Since logistic does not touch the natural world, and since every one admits that mathematics does give us truth, the only possibility left to Russell was to assert the existence of a suprasensible world, the world of universals of Plato, in another form. In mathematics, he says, we are studying this world and making discoveries in it. It exists outside of the existence of any individual mind, and its laws are the laws of logistic naturally. That such world exists we will readily admit, yet we deny that it stands finished as a Greek temple in all

its cold and austere beauty, but that it is rather a living organism similar to the earth in geologic times, and out of the stress of temperature and moisture and dazzling sun there is evolved through the ages a succession of increasingly intricate and complex forms. But these forms derive their existence from the push and surge of the human mind beating against the cliffs of the unknown. Even logistic itself is the outburst of the mind from the barriers of the early attempts to think and to think clearly. Mathematics finally attacked even the process of thinking, just as it had considered number, space, operations, and hypernumber, and created for itself a more active logic. That this should happen was inevitable. Says Brunschvicg:[8]

"Symbolic logic, like poetic art following the spontaneous works of genius, simply celebrates the victory or records the defeat. Consequently it is upon the territory of positive science that the positive philosophy of mathematics should be placed. It gives up the chimerical ideal of founding mathematics upon the prolongation beyond the limits imposed by methodical verification itself of the apparatus of definitions, postulates, and demonstrations; it becomes immanent in science with the intention of discerning what is incorporated therein of intelligence and truth."

The philosophic assumption at the root of the view taken by the supporters of logistic as the sole source of truth we are not much concerned with, since we are not discussing philosophy but mathematics. But we may inspect it a little with profit. This assumption is the very old one, that there is an absolute truth independent of human existence and that by searching we may find it out. Says Jourdain:[9]

"At last, then, we arrive at seeing that the nature of

[8] *Les étapes de la philosophie mathématique,* p. 426.
[9] *Nature of Mathematics,* p. 88.

mathematics is independent of us personally and of the world outside, and we can feel that our own discoveries and views do not affect the truth itself, but only the extent to which we or others can see it. Some of us discover things in science, but we do not really create anything in science any more than Columbus created America. Common sense certainly leads us astray when we try to use it for purposes for which it is not particularly adapted, just as we may cut ourselves and not our beards if we try to shave with a carving knife; but it has the merit of finding no difficulty in agreeing with those philosophers who have succeeded in satisfying themselves of the truth and position of mathematics. Some philosophers have reached the startling conclusion that truth is made by men, and that mathematics is created by mathematicians, and that Columbus created America; but common sense, it is refreshing to think, is at any rate above being flattered by philosophical persuasion that it really occupies a place sometimes reserved for an even more Sacred Being."

Doubtless if Columbus were to discover America over again he might conclude that acts of creation had gone on in the meantime, and might reasonably assume that they happened in the past, and doubtless Mr. Jourdain is forced to conclude from his own argument that the words he uses in the English tongue have not been built up by the efforts of man but have existed from the beginnings of time, that the idea of propositional function and of relative and of function, pointset, transfinite number, Lobatchevskian space, and a long list of other terms, have always been waiting in the mines of thought for the lucky prospector, but common sense would refute this view with very little study of the case. We may grant that electric waves have always existed, but that the wireless telegraph has always existed in any sense is not true; nor that even if carbon, nitrogen, hydrogen and oxygen have always existed, nitroglycerine

is to be dug out of wells, or that because sound-waves exist in the air, that therefore symphonies, operas, and all music have always been waiting to be discovered, not created. It is true perhaps that the elementary units out of which things material or mental are constructed exist in some sense, external to any one individual in some sense, but it is not true that therefore the combinations of these elements have always existed. Logistic, with all its boasted power, has never constructed a theorem that was truly synthetic in character, it has never taken a set of new postulates not derived from previously existing theories and developed a branch of mathematics similar to geometry or an algebra. It is powerless to move without the constant attendance of the intellect, it draws no more conclusions than Jevons's logical machine without its operator. It has never even introduced as one of its results a new thought of wide-reaching power, such as the idea of propositional function itself. This idea came from the extension of the mathematical function to other things than quantity. Columbus did not create the trees nor Indians nor shores of America, but he did create something that the Icelanders or the Chinese or other reputed previous discoverers did not create, and its existence we celebrate to-day more than the forgotten Indians, or the shifting sands of Watling's Island, or the broken tree-trunks. Mathematics, as we said before, did not spring like Athena from the head of Zeus, nor is it the record of the intellectual microscope and scalpel, but rather as Pringsheim,[10] who is not a philosopher but a mathematician, says: "The true mathematician is always a good deal of an artist, an architect, yes, a poet. Beyond the real world, though perceptibly connected with it, mathematicians have created an ideal world which they attempt to develop into the most perfect of all worlds, and which is being explored in every direction. None has the faintest conception of this world

[10] *Jahr. Deut. Math. Ver.*, Vol. XXXII, p. 381.

except him who knows it; only presumptuous ignorance can assert that the mathematician moves in a narrow circle. The truth which he seeks is, to be sure, broadly considered, neither more nor less than consistency; but does not his mastership show indeed in this very limitation? To solve questions of this kind he passes unenviously over others."

We must pass on however to the reef that wrecked logistic in its short voyage after imperial dominion. This is nothing less than infinity itself. Since logistic asserted philosophically the suprasensible and supramental existence of its objects, it was forced to assert that there is an absolute infinity. In the transfinites of Cantor it found ultimately its ruin. In order to handle classes that had an infinite number of members it had to set up definitions that ultimately led to the contradictions which in the *Principles of Mathematics* of Russell were left unsolved. These were the objects of the assaults of Poincaré and others, and led to the definitive abandonment of the second volume of the *Principles*. The presentation of the *Principia* has many modifications, too long to cite, but the discussions in the *Revue de métaphysique et de morale* from 1900 on will be found very illuminating in their bearing on the nature of mathematics. The philosophical writings of Poincaré particularly should be consulted. The net result of all the discussions is that all the metaphysics has been eliminated from logistic, and it assumes its proper place in the mathematical family, as a branch of mathematics on a par with the other branches, such as arithmetic, geometry, algebra, group-theory.

The question of infinity is one of the most difficult to consider, and in one of his last articles Poincaré despairs of mathematicians ever agreeing upon it. The reason for perpetual disagreement he gives is the fundamental difference in point of view of reasoning in general. If the objects of mathematics are supramental, then the mind is forced

to admit an absolute infinity. If the objects of mathematics are created by the mind, then we must deny the absolute infinity. So far no decisive criterion has appeared, beyond that laid down by Poincaré, that any object about which we talk or reason must be defined, that is, made to be distinguishable from all other objects, in a finite number of words. For example, there is no such thing as the collection of all integers, since while we may define the class of integers and also any one integer we cannot define each and every integer. When logistic seeks to correlate the collection of all integers to any other infinite collection, member to member, this criterion demands that a law of correlation be stated which may be applied to every member of the class. This is manifestly impossible. A case is the proof that rational numbers may be put into a one-to-one correspondence with the integers. While any one rational may be placed in this way, or any finite number of them, yet according to the criterion it is not possible to decide that we can place every rational in this way. Manifestly any operation that has to be done in successive steps will never reach an absolute infinity. All proofs relating to infinite collections consider that the statement of a law for any member of the class is sufficient. The criterion demands a law for every member, which is admittedly not possible. The absolute infinity must not be confused with the mathematical infinity, which is merely an unlimited or arbitrary class. In all the processes we use in getting limits, the infinity that enters is not the Cantor infinity.

We may then safely conclude that logistic furnishes truth to the other branches of mathematics in much the same way that algebra does to geometry, or geometry to algebra, or numbers to group-theory, or hypernumbers to geometry. By logistic we may draw conclusions about the elements with which we deal. If we try to interpret the

conclusions logistic is powerless to do so any more than geometry can yield us theorems in logic. Also the processes in reasoning of any kind are no different in logistic from what they are in algebra, geometry, theory of numbers, theory of groups, and it is the intelligence, not the logistic, that draws the conclusion of logistic, just as it is the mathematician that solves algebraic equations, not algebra. Logistic has a right therefore to exist as an independent branch of mathematics, but it is not the Overlord of the mathematical world. As to the philosophical import of logistic, we may well follow Poincaré's advice, and continue the development of mathematics with little concern whether realism or idealism or positivism is substantiated in the philosophical world. Indeed we may conclude eventually with Lord Kelvin[11] that "mathematics is the only true metaphysics."

References. Brunschvicg, *Les étapes de la philosophie mathématique.*
Hadamard, "La logistique et la notion de nombre entier." *Rev. gén. des sci.,* XVI (1906), pp. 906-914.
Keyser, "The Thesis of Modern Logistic." *Science,* Vol. XXX (1909), pp. 949-963.
Moore, "On the Foundations of Mathematics." *Bull. Amer. Math. Soc.,* Ser. 2, Vol. IX (1903), p. 402.
Hobson, "On the Infinite and Infinitesimal in Mathematical Analysis." *Proc. Lond. Math. Soc.,* Vol. XXXV (1903), pp. 117-140.

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[11] *Life,* p. 10.